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Fractional Complex Transforms for Fractional Differential Equations

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Abstract— The aim of this paper is by using the fractional complex transform and the optimal homotopy analysis by method (OHAM) to find the analytical approximate solutions for nonlinear partial fractional differential Zakharov-Kuznetsov equation. Fractional complex transformation is proposed to convert nonlinear partial fractional differential Zakharov-Kuznetsov equation to nonlinear partial differential equations. Also, we use the optimal homotopy analysis method (OHAM) to solve the obtained nonlinear PDEs. This optimal approach has general meaning and can be used to get the fast convergent series solution of the different type of nonlinear partial fractional differential equations. The results reveal that this method is very effective and powerful to obtain the approximate solutions.

.Index Terms— Homotopy analysis method; Optimal value; Fractional complex transform; fractional Zakharov-Kuznetsov equations.

1 INTRODUCTION

THE transform method is an important method to solve mathematical problems. Many useful transforms for solving various problems appeared in the literature, such as the traveling wave transform, the Laplace transform, the Fourier transform, other classical integral transforms, and the local fractional integral transforms (see [1]). Recently, it was suggested to convert fractional order differential equations with local fractional derivative, and the resultant equations can be solved by some advanced calculus [2]. The fractional complex transform was first proposed in Refs. [3] and [4].

Mohamed S. Mohamed et al.[5] used the fractional complex transformation to obtain the exact solutions for some nonlinear partial fractional differential equations. There are many methods for obtaining analytic approximate solutions for nonlinear fractional differential equations.

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Our objective is to obtain analytical solutions of the following time and space fractional derivatives nonlinear differential equations by Fractional Complex Transform (FCT) with the help of OHAM, and to determine the effectiveness of FCT in solving these kinds of problems. Gepreel et al. [6] have used the complex transformation to calculate the exact solution for some nonlinear fractional differential equations.

In this paper, we consider the nonlinear fractional Zakharov Kuznetsov equation [7]. The nonlinear fractional Zakharov-Kuznetsov equation (FZK) is taking the following form:.

$$D_t^{\alpha} u + a\,(u^{p})_{x} + b\,(u^{q})_{xxx} + c\,(u^{r})_{yyx} = 0. \qquad (1.1)$$

where $u = u(x;y;t)$, is a parameter describing the order of the fractional derivative , a ,b and c are arbitrary constants and p , , and r are non-zero integers.

The aim of this paper is to utilize the fractional complex transform (FCT) with the aid of HAM to find the approximate solution for nonlinear fractional Zakharov Kuznetsov equation. In addition, we would like to determine the effectiveness of FCT in solving these kinds of problems.

2 PRELIMINARIES AND NOTATIONS

In this section, we mention some basic definitions of fractional calculus theory which can be used further in this work. Local fractional derivative to order in interval is defined by [8] and [9]

(2.1)

where $\Delta^{\alpha}(f(x)-f(x_0))=\Gamma(\alpha+1)\Delta(f(x)-f(x_0))$. Also, the inverse of local fractional derivative to of order in interval [a,b] is defined by [8] and [10] as follows:-

(2.2)

where , are the partition of the interval [a,b].

3 The Optimal Homotopy Analysis Method

For more clarifications about these basic ideas of the OHAM for nonlinear partial differential equations. It's better to see the following nonlinear partial differential equation:

(3.1)

where N is a nonlinear operator for this problem, x and t denotes the independent variables, and is an unknown function:

Through using the HAM, we first construct zero-order deformation equation

(3.2)

where is the embedding parameter, is an auxiliary parameter, is an auxiliary function, is an auxiliary linear operator, is an initial guess, at and , we have

(3.3)

By considering a Taylor series expression of with respect to in the form

(3.4)

where

(3.5)

the initial guess, the auxiliary parameter h and the auxiliary function $H(t)$ are selected such that the series is convergent at , then we have from

(3.6)

We give the definition of the vector

(3.7)

Differentiating m times with respect to , then setting and dividing then by , we have the - order deformation equation

(3.8)

where

(3.9)

and

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called optimization method to find out the optimal convergence control parameters by minimum of the square residual error integrated in the whole region having physical meaning. Their method depends on the square residual error. Let denote the square residual error of the governing equation (3.1) and express as

(3.12)

where

(3.13)

the optimal value of h is given by a nonlinear algebraic equation as:

(3.14)

4 The Fractional Complex Transform

The following nonlinear partial fractional differential equation is given,

(4.1)

where

,

and

denote the Local fractional derivative with respect to respectively. The fractional complex transform requires that

(4.2)

where , and are unknown constants. Using the basic properties of the fractional derivative and the above transforms, we can convert fractional derivatives into the following clasical partial derivatives:

Therefore, we can easily convert the nonlinear partial fractional differential equations into the nonlinear partial differential equations which can be solved by using the optimal homotopy analysis.

5 Equations

We consider the time-fractional FZK(3, 3, 3) in the form:

(5.1)

where $0 < \alpha \leq 1$ is a parameter describing the order of the fractional time derivative. The exact solution to eq. (5.1) when $\alpha = 1$ and subject to the initial condition

(5.2)

where ρ is an arbitrary constants was derived in [6] and is given as:

(5.3)

To apply FCT to eq. (5.1), we use the above transformations, so we have the following partial differential equation:

(5.4)

For simplicity we set so we get

(5.5)

Now, we solve eq. (5.5) by means of HAM, we choose the linear operator

(5.6)

with property where is a constant. We define a nonlinear operator as

(5.7)

We construct the zeroth-order deformation equation

For and , we can write

(5.8)

Thus, we obtain the -order deformation equations

(5.9)

where

Now the solution of the -order deformation equations (5.9) for $m \geq 1$ and $H(t) = 1.$ become

(5.10)

So, a few terms of series solution are as follows:

(5.11)

(5.12)

According to the HAM, we can conclude that

(5.13)

Therefore, substituting the values of $u_0(x, y, t)$ and from Eqs. ((5.11), ((5.12) into. Eq.(5.13) yields:

(5.14)

When and we obtain

(5.15)

which the same as the solution obtained by [7]. Then we find at $h = -1$ and

To investigate the influence of on the convergence of the solution series given by the HAM, we first plot the so-called h -curves of $u''(0,0,1)$. According to the h -curves, it is easy to discover the valid region of h . We used terms in evaluating the approximate solution $u(x, y, t) = \sum_{i=0}^{4} u_i(x, y, t)$. Note that the solution series contains the auxiliary parameter h which provides us with a simple way to adjust and control the convergence of the solution series. In general, by means of the socalled -curve i.e., a curve of a versus h . As pointed by Liao [15] and Mohamed S. Mohamed et al. [16-17] , the valid region of h is a horizontal line segment. Therefore, it is straightforward to choose an appropriate range for h which ensure the convergence of the solution series. We stretch the h -curve of $u''(0, 0, 1))$ in Fig. 1, which shows that the solution series is convergent when $-0.2 \leq h \leq 0.2$ and $-1.2 \leq h \leq 0.5.$

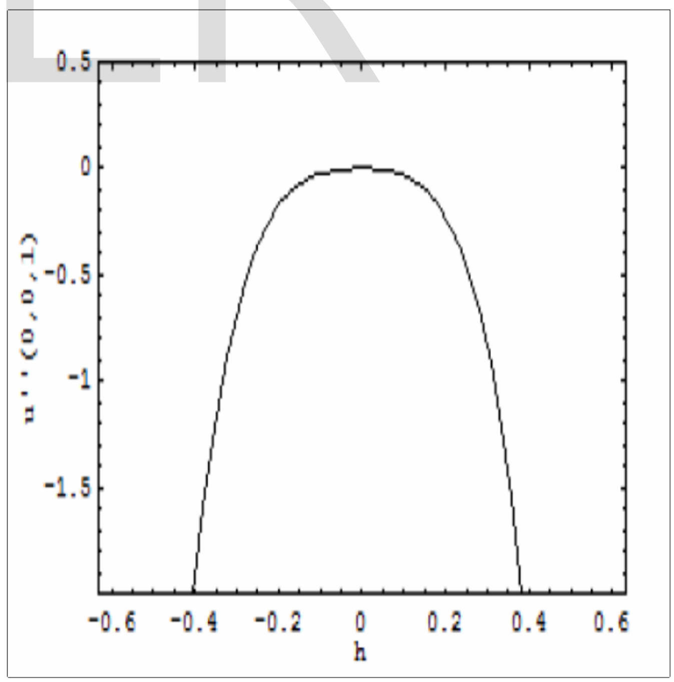

Fig.1 The h-curve of $u''(0,0,1)$ at the 5th order of approximation when $H(x,y,t)=1$, $=1$ and $=0.1$

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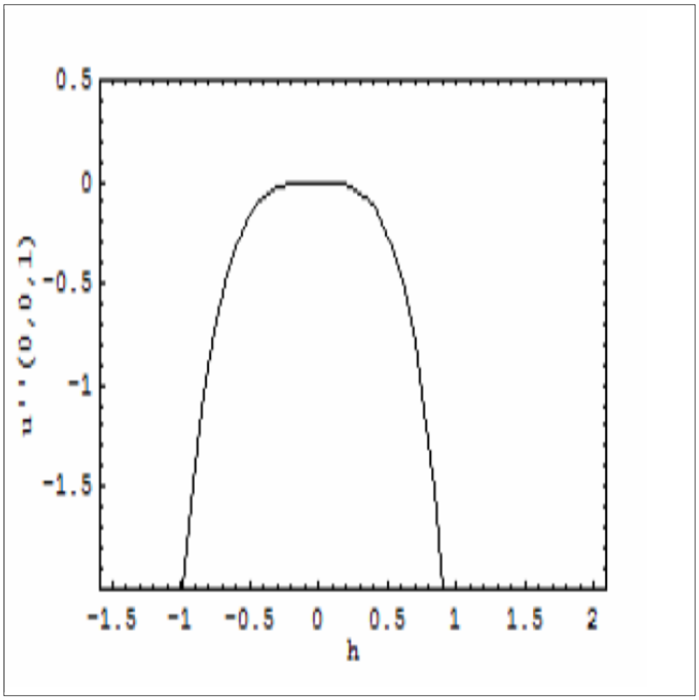

Fig.2. The *h*-curve of $u''(0,0,1)$ at the 5*th* order of approximation when $H(x,y,t)=1$, $\alpha=1$ and $\rho=0.05$

$x = 0.2, y = 0.2$ and $t \in [-1.2, 0.5]$			
α	ρ	Optimal value of h	Minimum value of E_N
1.0	0.1	-0.3	1.64986×10^{-8}
	0.05	-0.990321	1.37312×10^{-22}
	0.005	-0.995339	7.9675×10^{-29}

Table 1: The optimal values of h.

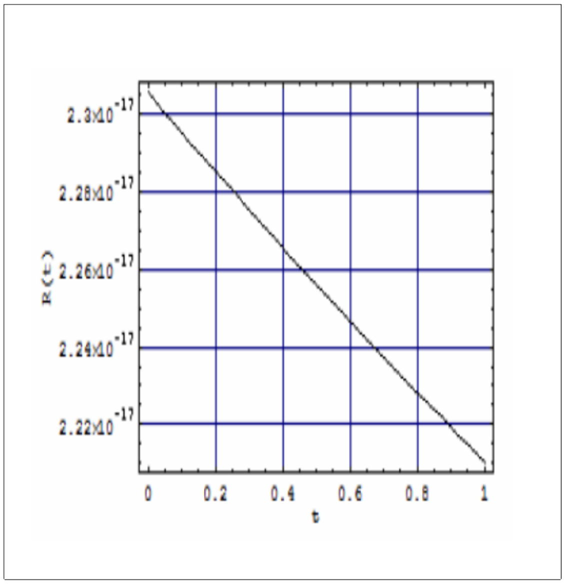

Fig. 5. The residual of the 5th order approximation for $h = -0.995$, $\gamma = 0.005$, $x = 0.2$, $y = 0.2$ and $\alpha = 1$

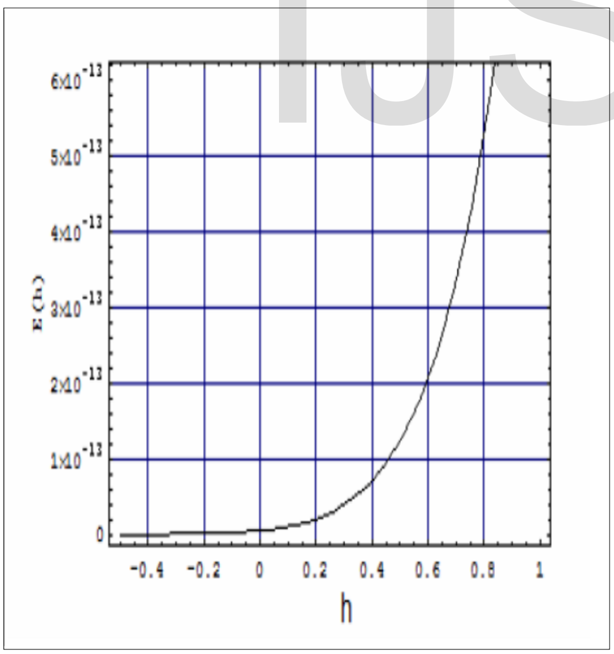

Fig. 6. The Square residual error for 5th order approximation for $h = -0.995$, $\gamma = 0.005$, $x = 0.2$, $y = 0.2$ and $\alpha = 1$

5 CONCLUSION

In this paper, the fractional complex transform is very simple and use of this method does not need the knowledge of fractional calculus. The fractional complex transform is the simplest approach; it is to convert the fractional differential equations into ordinary differential equations, making the solution procedure extremely simple. Recently, the fractional complex transform has been suggested to convert fractional-order differential equations with modified Riemann-Liouville derivatives into integer order differential equations, and the reduced equations can be solved by symbolic computation. OHAM has been successfully applied to obtain the numerical solutions of the nonlinear partial fractional differential Zakharov-Kuznetsov equation was initial conditions. The reliability of this method and reduction in computations give this method a wider applicability. OHAM contains a certain auxiliary parameter which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. OHAM is clearly a very efficient and powerful technique for finding the numerical solutions of the proposed equation.

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